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THE STOKES AND KRASOVSKII CONJECTURES FOR THE WAVE OF GREATEST --ETC(U)
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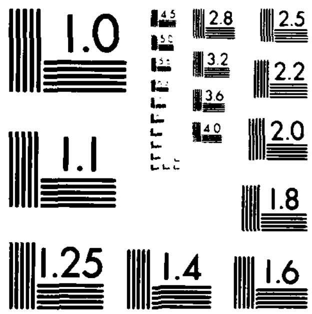

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THE STOKES AND KRASOVSKII CONJECTURES FOR THE WAVE OF GREATEST HEIGHT

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ABSTRACT

The integral equation

$$\phi_\mu(s) = \frac{1}{3\pi}\int_0^\pi \frac{\sin\phi_\mu(t)}{\mu^{-1}+\int_0^t \sin\phi_\mu(u)\,du} \log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| dt$$

was derived by Nekrasov to describe waves of permanent form on the surface of a non-viscous, irrotational, infinitely deep flow, the function ϕ_μ giving the angle which the wave surface makes with the horizontal. The wave of greatest height is the singular case $\mu = \infty$, and it is shown that there exists a solution ϕ_∞ to the equation in this case and that it can be obtained as the limit (in a specified sense) as $\mu \to \infty$ of solutions for finite μ .

Stokes conjectured that $\phi_\infty(s) \to \frac{1}{6}\pi$ as $s \downarrow 0$, so that the wave is sharply crested in the limit case; and Krasovskii conjectured that $\sup_{s\in[0,\pi]} \phi_\mu(s) \le \frac{1}{6}\pi$ for all finite μ. While the present paper makes only limited progress towards deciding Stokes' conjecture, Krasovskii's conjecture is shown to be false for sufficiently large μ, the angle exceeding $\frac{1}{6}\pi$ in what is a boundary layer.

AMS(MOS) Subject Classification: 45G05, 45G10, 76B15, 76B25

Key Words: Water waves, periodic waves, waves of permanent form, free boundary problems, Stokes' conjecture, Nekrasov integral equation, singular perturbations, boundary layer

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SIGNIFICANCE AND EXPLANATION

It is shown that there exists a solution to Nekrasov's integral equation which describes a wave of greatest height and of permanent form moving on the surface of a non-viscous, irrotational, infinitely deep flow. It is also shown that this wave can be obtained as the limit, in a specified sense, of waves of almost extreme form.

Stokes conjectured, almost 100 years ago, that in the extreme case the wave is sharply crested and the wave surface makes an angle of $\frac{1}{6}\pi$ with the horizontal at the crest, and Krasovskii conjectured that, for waves of non-extreme form, which are smooth-crested, the angle between the surface and the horizontal at no point exceeds $\frac{1}{6}\pi$, the latter belief being widely held until some recent numerical calculations cast some doubt upon it. While the present paper makes only partial progress towards deciding Stokes' conjecture, it does confirm the numerical evidence and prove that the Krasovskii conjecture is false for waves sufficiently close to the extreme form, the angle exceeding $\frac{1}{6}\pi$ in a boundary layer.

THE STOKES AND KRASOVSKII CONJECTURES

FOR THE WAVE OF GREATEST HEIGHT

J. B. McLeod

1. Introduction

This paper considers the problem of a wave of constant periodic form moving with constant velocity on the surface of a non-viscous fluid which is either of infinite depth or on a horizontal bottom. The motion is two-dimensional, i.e. the motion is independent of the coordinate in the horizontal direction perpendicular to the velocity of the wave, and if we restrict ourselves to irrotational flow and assume that the periodic form of the wave is in addition symmetrical about a vertical axis through a crest, then it is known that the shape of the wave can be described (in the case of infinite depth) by a solution of the equation

$$\phi(s) = \frac{1}{3\pi}\int_{-\pi}^{\pi} \frac{\sin\phi(t)}{\mu^{-1}+\int_0^t \sin\phi(u)\,du} \left\{\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k}\right\} dt \ . \tag{1.1}$$

This equation is due to Nekrasov [1]. An exposition of its deduction can be found in [2], and in [3] an analysis of the equivalence between (1.1) and other formulations of the problem, which we shall not however require in the present paper. The equation is obtained by mapping the region under one wave-length (from trough to trough) conformally onto the unit disc cut along the negative real axis. The generic point on the circumference of the disc is e^{is}, with $-\pi < s < \pi$, and $\phi(s)$ gives the angle between the wave surface and the horizontal at the point on the surface which corresponds to the point e^{is} on the circumference of the disc. The constant μ is given by

$$\mu = \frac{3g\lambda c}{2\pi Q^3},$$

where g is the acceleration due to gravity, λ the wave-length of the periodic wave, c the speed at which the wave form is progressing, and Q the speed of particles at the crest of the wave. In obtaining (1.1) it is assumed (as we have already mentioned) that the wave is symmetrical about a vertical axis through a crest, and this is reflected in the fact that (1.1) certainly implies that $\phi(-s) = -\phi(s)$. Using this

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we can restrict attention to the interval $[0,\pi]$ and take the equation in the form

$$(1.2) \qquad \phi(s) = \frac{2}{3\pi}\int_0^\pi \frac{\sin\phi(t)}{\mu^{-1} + \int_0^t \sin\phi(u)du} \left\{ \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \right\} dt ,$$

or, after summation of the series,

$$(1.3) \qquad \phi(s) = \frac{1}{3\pi}\int_0^\pi \frac{\sin\phi(t)}{\mu^{-1} + \int_0^t \sin\phi(u)du} \log\left| \frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)} \right| dt,$$

and the last form is the form in which we shall mainly consider it.

For a fluid of finite depth there is a comparable formula; with the same interpretations on ϕ and s, we have

$$(1.4) \qquad \phi(s) = \frac{1}{3\pi}\int_0^\pi \frac{\sin\phi(t)}{\mu^{-1} + \int_0^t \sin\phi(u)du} \log\left| \frac{\mathrm{sn}\{\pi^{-1}K(s+t)\}}{\mathrm{sn}\{\pi^{-1}K(s-t)\}} \right| dt ,$$

where sn denotes the Jacobian elliptic function whose quarter periods K, iK' satisfy

$$K'/K = 4h/\lambda ,$$

h being the mean depth of the fluid.

Nekrasov himself discussed solutions of (1.3) and (1.4) for waves of small amplitude, but the first to tackle successfully the question of waves whose amplitude is not necessarily small was Krasovskii [4]. Using a different but equivalent form of (1.3-4) (see [5] for an exposition of this equivalence), Krosovskii showed that, for each β with $0 < \beta < \frac{1}{6}\pi$, there exists a corresponding value of μ and a continuous solution ϕ of (1.3) (or (1.4)) such that $\phi \geq 0$ and

$$\sup_{s\in[0,\pi]} \phi(s) = \beta.$$

The method is essentially a degree theory argument in which the inequality $\beta < \frac{1}{6}\pi$ plays a crucial role, but the approach does not give the range of values of μ for which the solution exists. Krasovskii's solutions all satisfy $\phi(0) = \phi(\pi) = 0$, as indeed (1.3-4) imply if μ is finite, and so represent smooth-crested waves.

The gap over the range of values of μ was filled by Keady and Norbury [3], who have shown, again by degree theory arguments, that one can find a solution of (1.3) bifurcating from the trivial solution at the first eigenvalue $\mu = 3$ of the linearised problem, and then follow it continuously for all finite μ. Their final result is that, for all finite $\mu > 3$, there exists a continuous solution ϕ of (1.3) such that ϕ is not identically zero and $0 \le \phi < \frac{1}{2}\pi$. In the case of (1.4), the result remains true with $\mu > 3$ replaced by $\mu > 3 \coth(2\pi h/\lambda)$. (It is known that there can be no solution ϕ with these properties if $0 < \mu \le 3$ in the case of (1.3) or $0 < \mu \le 3 \coth(2\pi h/\lambda)$ in the case of (1.4).) Again, the Keady-Norbury waves are smooth-crested.

The case $\mu = \infty$ $(Q = 0)$ corresponds to the presence of a stagnation point at the wave crest, and it is the case in which, for given c, the wave reaches the greatest height above mean level [6]. In 1880 Stokes [7] conjectured that there does indeed exist a wave in this limiting case, but that it is peaked instead of smooth-crested, and he argued, on the basis of an asymptotic approximation near the crest, that for the corresponding solution of (1.3-4)

(1.5) $$\lim_{s \downarrow 0} \phi(s) = \frac{1}{6}\pi ,$$

i.e. that at the peak the slope of the wave is inclined at $\frac{1}{6}\pi$ to the horizontal. It is not difficult to show that if there exists a solution ϕ to (1.3) (or (1.4)) with $\mu = \infty$, and if that solution (assumed continuous on $(0,\pi]$ with $0 \le \phi \le \frac{1}{2}\pi$) is sufficiently regular near the origin that $\lim_{s \downarrow 0} \phi(s)$ exists and is non-zero, then necessarily (1.5) holds. Toland [5] gives a proof, and for completeness another (perhaps simpler) is given in §2 below. But the difficulty is to establish first that there is indeed a solution, and secondly that the solution has sufficient regularity.

The obvious approach is to take the Keady-Norbury solution for finite μ, and show that it converges to a solution of the limit equation as $\mu \to \infty$, at least through some sequence of values. In [5] Toland carries through this process, using some rather deep results from the theory of Fourier series, and concludes that there is convergence to a solution of the limit equation, but he can prove effectively no regularity

properties near the crest, so that (1.5) remains unproved. Toland works always with (1.3) but he remarks that the method extends to (1.4).

The first aim of the present paper is to give a quite different account of the convergence process from that given by Toland. The method uses little more than elementary manipulations with the integral equation, and is both simpler than Toland's and stronger, in that more detailed information is obtained. Even this more detailed information is however insufficient to decide the truth of (1.5).

We work throughout with (1.3), but the argument is essentially unchanged for (1.4), as we point out. Our goal therefore is the following theorem.

Theorem 1. *If* $\mu = \infty$, *there exists for* $s > 0$ *a solution* $\phi(s)$ *of (1.3) with the following properties*:

(i) ϕ *is continuous on* $(0,\pi]$;

(ii) $0 \leq \phi \leq \frac{1}{2}\pi$;

(iii) $\phi(s)$ *is bounded from zero as* $s \downarrow 0$;

(iv) ϕ *is the limit of a sequence of functions* $\{\phi_\mu\}$ *as* $\mu \to \infty$, *where* ϕ_μ *is a non-trivial solution of (1.3) continuous on* $[0,\pi]$ *and satisfying* $0 \leq \phi_\mu < \frac{1}{2}\pi$. *This limit process is uniform on* $[\eta,\pi]$ *for any fixed* η *with* $0 < \eta < \pi$.

Theorem 2. *Theorem 1 remains valid if (1.3) is replaced by (1.4).*

Remarks. 1. In §3 we reduce the proof of Theorem 1 to that of two lemmas, which are then proved in the succeeding sections.

2. The proof of Theorem 2, as we have already mentioned, is almost identical with that of Theorem 1. What little needs to be said is said in a short section at the end of the proof of Theorem 1.

The equation (1.5) embodies what is conventionally regarded as "Stokes' conjecture". But in fact, in his paper in 1880, Stokes says rather more. Having made the conjecture, he goes on as follows.

"But whether in the limiting form the inclination of the wave to the horizon continually increases from the trough to the summit, and is consequently limited to 30°,

or whether on the other hand the points of inflexion which the profile presents in the general case remain at a finite distance from the summit when the limiting form is reached, so that on passing from the trough to the summit the inclination attains a maximum from which it begins to decrease before the summit is reached, is a question which I cannot certainly decide, though I feel little doubt that the former alternative represents the truth."

More briefly, Stokes is making the further conjecture that the limiting solution ϕ satisfies $\phi' < 0$. I suspect that the proof of this second conjecture is even more difficult than that of the first.

Stokes, however, has not been the only one to make conjectures about this problem. Krasovskii, in the light of his work in [4], was led to two conjectures which, expressed in our notation, are as follows.

1. When $\sup\limits_{s \in [0,\pi]} \phi_\mu(s)$ tends to $\frac{1}{6}\pi$, the solution ϕ_μ tends to the limit solution with $\mu = \infty$.

2. There exists no solution ϕ_μ with $\sup\limits_{s \in [0,\pi]} \phi_\mu(s) > \frac{1}{6}\pi$.

The truth of these conjectures is now in some doubt because of recent numerical evidence by Longuet-Higgins and Fox [8]. The numerical results indicate that, once μ is sufficiently large, $\sup\limits_{s \in [0,\pi]} \phi_\mu(s)$ does slightly exceed $\frac{1}{6}\pi$, by .37°, although it does so in the boundary layer, i.e., at values of s which tend to zero as $\mu \to \infty$ so that the effect dies out in the limit case. Our extimates enable us to make an examination of the behaviour of the boundary layer and give an analytical proof that Krasovskii's conjectures are indeed false.

<u>Theorem 3</u>. <u>The sequence of functions</u> $\{\phi_\mu\}$ <u>in Theorem 1 or in Theorem 2 must satisfy</u> $\sup\limits_{s \in [0,\pi]} \phi_\mu(s) > \frac{1}{6}\pi$ <u>if</u> μ <u>is sufficiently large</u>.

The proof, which is given in the final sections of the paper, is a matter of showing that in the boundary layer the function ϕ_μ (with its argument suitably scaled) tends as $\mu \to \infty$ to a solution of the integral equation

$$(1.6) \qquad \phi(s) = \frac{1}{3\pi} \int_0^{\infty} \frac{\sin\phi(t)}{1+\int_0^t \sin\phi(u)\,du} \log\left|\frac{s+t}{s-t}\right| dt ,$$

and then investigating the asymptotic behaviour of solutions of (1.6) as $s \to \infty$. It is a natural question to ask whether the number of roots of $\phi_\mu = \frac{1}{6}\pi$ becomes unboundedly large as $\mu \to \infty$, and the answer to this is presumably in the affirmative. But the theorem states only that there is at least one solution for μ sufficiently large, and as is noted at the end of the proof of the theorem, to prove more would seem to entail an altogether more detailed examination of the asymptotics of (1.6) and is therefore not attempted in this paper.

2. A formal proof of (1.5)

Our object is to prove that if ϕ is a solution of

$$\phi(s) = \frac{1}{3\pi}\int_0^\pi \frac{\sin\phi(t)}{\int_0^t \sin\phi(u)\,du}\,\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\,dt$$

which is continuous on $(0,\pi]$ with $0 \le \phi \le \frac{1}{2}\pi$, and if

(2.1) $$\lim_{s\to 0} \phi(s) = \lambda \neq 0 ,$$

then necessarily $\lambda = \frac{1}{6}\pi$. (An almost identical proof, which we shall not give, applies to (1.4) with $\mu = \infty$.)

In view of (2.1), we have

$$\frac{\sin\phi(t)}{\int_0^t \sin\phi(u)\,du} \sim \frac{1}{t} \quad \text{as } t \downarrow 0 ,$$

and it is well known that

$$\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| = O\left(\frac{s}{t}\right)$$

if s is of smaller order than t, and that, if both s and t are small,

$$\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| \sim \log\left|\frac{s+t}{s-t}\right| .$$

Thus, for small s (> 0)

$$\phi(s) = \frac{1}{3\pi}\left\{\int_0^{s^{\frac{1}{2}}} + \int_{s^{\frac{1}{2}}}^{\pi}\right\}\frac{\sin\phi(t)}{\int_0^t \sin\phi(u)\,du}\,\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\,dt$$

$$\sim \frac{1}{3\pi}\int_0^{s^{\frac{1}{2}}} \frac{1}{t}\log\left|\frac{s+t}{s-t}\right|dt + O\left\{\int_{s^{\frac{1}{2}}}^{\pi} \frac{1}{t}\cdot\frac{s}{t}\,dt\right\}$$

$$= \frac{1}{3\pi}\int_0^{s^{-\frac{1}{2}}} \frac{1}{u}\log\left|\frac{1+u}{1-u}\right|du + O(s^{\frac{1}{2}}) ,$$

by making the transformation $t = su$ in the first integral. But the integral in the last line clearly tends (as $s \downarrow 0$) to

(2.2) $$\int_0^\infty \frac{1}{u}\log\left|\frac{1+u}{1-u}\right|du ,$$

and since the value of this last integral is $\frac{1}{2}\pi^2$, the result (1.5) follows. (The integral (2.2) can be evaluated, for example, by noting that the contribution to the integral from [0,1] is equal to the contribution from [1,∞], as is seen by the transformation $u \leftrightarrow u^{-1}$, and then evaluating the integral over [0,1] by expanding the integrand in a power series and using $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$.)

3. The proof of Theorem 1

In the proof of Theorem 1 and the attendant lemmas, ϕ_μ will be a non-trivial solution of (1.3), for finite $\mu(>3)$, with ϕ_μ continuous on $[0,\pi]$ and $0 \le \phi_\mu \le \frac{1}{2}\pi$, $\phi_\mu(0) = \phi_\mu(\pi) = 0$. The existence of ϕ_μ is guaranteed by the work of Keady and Norbury.

The letter K will stand for various positive constants, not necessarily the same at each appearance, but always independent of any of the parameters under consideration. The notation $K(\eta_1,\eta_2,\ldots,\eta_n)$ will mean that the constant K depends on the quantities $\eta_1,\ldots,\eta_n$, but on no other parameters in the problem.

The first step is to obtain an estimate for the denominator in the integrand in (1.3) as $\mu \to \infty$. This is the effect of Lemma 1, which is proved in §4 below.

<u>Lemma 1</u>. $\mu^{-1} + \int_0^\eta \sin\phi_\mu(u)du \ge K\,\eta$, <u>where the positive constant</u> K <u>is independent of both</u> μ <u>and</u> η.

We also have (proved in §5 below)

<u>Lemma 2.</u> <u>The functions</u> ϕ_μ <u>are equicontinuous in</u> $[\eta,\pi]$ <u>for any fixed</u> η <u>with</u> $0 < \eta < \pi$.

Lemma 2, together with the bounds $0 \le \phi_\mu < \frac{1}{2}\pi$, enables us to apply the Ascoli-Arzelà theorem in any fixed interval $[\eta,\pi]$, and to conclude that there must be some sequence $\{\phi_\mu\}$ which is pointwise convergent on $(0,\pi]$ as $\mu \to \infty$ and uniformly so on $[\eta,\pi]$. The limit ϕ is of course continuous on $(0,\pi]$ and satisfies $0 \le \phi \le \frac{1}{2}\pi$, and by applying the dominated convergence theorem to (1.3), with the integrand bounded by

$$\frac{K}{t}\log\left|\frac{s+t}{s-t}\right| , \tag{3.1}$$

we see immediately that ϕ satisfies the limit equation, i.e. (1.3) with $\mu = \infty$. The proof of Theorem 1 is therefore complete once we have established that $\phi(s)$ is bounded from zero as $s \downarrow 0$.

To show this, note that $0 \le \phi \le \frac{1}{2}\pi$ implies that

$$\int_0^t \phi(u)du \le Kt \qquad \text{for } 0 \le t \le \pi ,$$

and from the equation (1.3) (with $\mu = \infty$) we have

$$\phi(s) \le K \int_0^{\frac{1}{2}s} \frac{\phi(t)}{t} \log \left| \frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)} \right| dt .$$

But for $0 \le t \le \frac{1}{2}s$ we have

(3.2) $$\log \left| \frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)} \right| \le K \frac{t}{s} ,$$

and so

$$\phi(s) \le \frac{K}{s} \int_0^{\frac{1}{2}s} \phi(t)dt \le K ,$$

the last inequality following by taking the limit in Lemma 1 as $\mu \to \infty$. This completes the proof of Theorem 1.

4. The proof of Lemma 1

If the lemma is proved for, say, $0 \le \eta \le \frac{1}{4}\pi$, then it is trivially true (with a possibly different K) for $0 \le \eta \le \pi$. We therefore assume $\eta \le \frac{1}{4}\pi$. Also $0 \le \theta \le \frac{1}{2}\pi$ implies

$$K_1 \le \sin\theta/\theta \le K_2 ,$$

and using this in (1.3), we have

$$\int_\eta^{2\eta} \frac{\phi_\mu(s)}{s}\,ds \ge K \int_\eta^{2\eta} \frac{\phi_\mu(t)}{\mu^{-1}+\int_0^t \phi_\mu(u)\,du} \left\{ \int_\eta^{2\eta} \frac{1}{s} \log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| ds \right\} dt.$$

For the relevant ranges of s,t,

$$K_1 \le \log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| \Big/ \log\left|\frac{s+t}{s-t}\right| \le K_2 , \tag{4.1}$$

and so, with $s = tv$,

$$\int_\eta^{2\eta} \frac{\phi_\mu(s)}{s}\,ds \ge K \int_\eta^{2\eta} \frac{\phi_\mu(t)}{\mu^{-1}+\int_0^t \phi_\mu(u)\,du} \left\{ \int_{\eta/t}^{2\eta/t} \frac{1}{v} \log\left|\frac{1+v}{1-v}\right| dv \right\} dt .$$

For the relevant values of t the inner integral is both bounded and bounded from zero, and so

$$\begin{aligned}\int_\eta^{2\eta} \frac{\phi_\mu(s)}{s}\,ds &\ge K\left[\log\left(\mu^{-1} + \int_0^t \phi_\mu(u)\,du\right)\right]_\eta^{2\eta} \\ &= K \log\left\{ 1 + \frac{\int_\eta^{2\eta} \phi_\mu(u)\,du}{\mu^{-1}+\int_0^\eta \phi_\mu(u)\,du} \right\} .\end{aligned}$$

Now the left-hand side is certainly bounded, since

$$\int_\eta^{2\eta} \frac{\phi_\mu(s)}{s}\,ds \le \frac{1}{2}\pi[\log s]_\eta^{2\eta} ,$$

and so the right-hand side is bounded. Also,

$$\log(1 + x) \ge Kx$$

for x positive and bounded. Hence

$$\begin{aligned}\int_\eta^{2\eta} \frac{\phi_\mu(s)}{s}\,ds &\ge K \frac{\int_\eta^{2\eta} \phi_\mu(u)\,du}{\mu^{-1}+\int_0^\eta \phi_\mu(u)\,du} \\ &\ge K \frac{\eta\int_\eta^{2\eta} \frac{\phi_\mu(u)}{u}\,du}{\mu^{-1}+\int_0^\eta \phi_\mu(u)\,du} ,\end{aligned}$$

from which the result of the lemma follows.

5. The proof of Lemma 2.

Let $s_1, s_2 \in [\eta, \pi]$. Without loss of generality we shall suppose $s_1 < s_2$, and we are interested in small values of $|s_1 - s_2|$. Then

$$\phi_\mu(s_1) - \phi_\mu(s_2) = \frac{1}{3\pi}\int_0^\pi \frac{t \sin\phi_\mu(t)}{\mu^{-1} + \int_0^t \sin\phi_\mu(u)\,du}\,\frac{1}{t}\left\{\log\left|\frac{\sin\frac{1}{2}(s_1+t)}{\sin\frac{1}{2}(s_1-t)}\right| - \log\left|\frac{\sin\frac{1}{2}(s_2+t)}{\sin\frac{1}{2}(s_2-t)}\right|\right\} dt$$

$$= I_1 + I_2 ,$$

say, where, for a given $\delta > 0$ (δ being thought of as being small compared with η but large compared with $|s_1 - s_2|$), I_2 is the integral over the part of $[0, \pi]$ lying in the interval $[s_1 - \delta, s_2 + \delta]$ and I_1 is the integral over the remainder of $[0, \pi]$. Since

$$\frac{d}{ds}\left\{\frac{1}{t}\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\right\} = \frac{1}{2t}\{\cot\tfrac{1}{2}(s+t) - \cot\tfrac{1}{2}(s-t)\}$$

(5.1)

$$= -\frac{\sin t}{2t\sin\frac{1}{2}(s+t)\,\sin\frac{1}{2}(s-t)}$$

it is clear, by use of Lemma 1, that in I_1 the integrand does not exceed $K(\eta)\delta^{-1}|s_1 - s_2|$, so that in fact

$$|I_1| \le K(\eta)\delta^{-1}|s_1 - s_2| ,$$

while

$$|I_2| \le K(\eta)\delta|\log\delta| .$$

The equicontinuity then follows by choosing first δ sufficiently small, and then $|s_1 - s_2|$. Specifically we could choose $\delta = |s_1 - s_2|^{\frac{1}{2}}$, which shows that actually the functions ϕ_μ are equi-Hölder-continuous for any exponent α with $\alpha < \frac{1}{2}$.

6. The proof of Theorem 2

The only difference from the proof of Theorem 1 is that the expression

$$\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| \tag{6.1}$$

has to be replaced by

$$\log\left|\frac{\mathrm{sn}\{\pi^{-1}K(s+t)\}}{\mathrm{sn}\{\pi^{-1}K(s-t)\}}\right| . \tag{6.2}$$

We have to verify only that the va. ous estimates used in connection with (6.1) apply equally well to (6.2). The specific places where these estimates appear are (3.1), (3.2), (4.1), (5.1), and there is no difficulty in carrying out the modifications at these points.

7. The proof of Theorem 3

We shall give the proof for the case of equation (1.3), leaving to the reader the very minor modifications necessary to deal with (1.4).

Once again we begin by stating two lemmas which are of independent interest and are proved in the succeeding sections. We obtain first an estimate on ϕ_μ' and ϕ', ϕ being the solution obtained in Theorem 1.

Lemma 3. *The functions* ϕ_μ *are continuously differentiable on* $[0,\pi]$ *and*

$$s|\phi_\mu'(s)| \le K \quad \text{for} \quad 0 \le s \le \pi ,$$

K *being independent of* μ. *Also,* ϕ *is continuously differentiable on* $(0,\pi]$, *and*

$$s|\phi'(s)| \le K \quad \text{for} \quad 0 < s \le \pi .$$

The next lemma asserts that the Stokes conjecture (1.5) is true at least in some average sense.

Lemma 4. $$\left|\int_\eta^{\frac{1}{2}\pi} \frac{\phi(s)-\frac{1}{6}\pi}{s}\, ds\right| \le K ,$$

where K *is independent of* η *as* $\eta \downarrow 0$.

We turn now to the behaviour in the boundary layer. From Lemma 1 we see that if η is of higher order than μ^{-1}, then in the expression

$$\mu^{-1} + \int_0^\eta \sin\phi_\mu(u)du$$

the integral term must dominate, while $0 \le \phi_\mu < \frac{1}{2}\pi$ implies that, if η is of smaller order than μ^{-1}, then μ^{-1} dominates. Since we certainly expect the integral to dominate outside any boundary layer, we are led to believe that the width of the boundary layer will be of order μ^{-1} and so to make the transformation

$$\sigma = \mu s , \qquad \phi_\mu^*(\sigma) = \phi_\mu(s) ,$$

and it is trivial to verify that ϕ_μ^* satisfies

$$\phi_\mu^*(\sigma) = \frac{1}{3\pi}\int_0^{\mu\pi} \frac{\sin\phi_\mu^*(\tau)}{1+\int_0^\tau \sin\phi_\mu^*(v)dv} \log\left|\frac{\sin\frac{1}{2}\mu^{-1}(\sigma+\tau)}{\sin\frac{1}{2}\mu^{-1}(\sigma-\tau)}\right| d\tau. \tag{7.1}$$

At the same time, the uniform bound $0 \le \phi_\mu^* < \frac{1}{2}\pi$ and the equicontinuity estimate $\sigma|\phi_\mu^{*\prime}(\sigma)| \le K$ which follows immediately from Lemma 3 assure us that, in any fixed interval $0 < k \le \sigma \le K < \infty$, there is a subsequence $\{\phi_\mu^*\}$ which converges uniformly

to ϕ^*, say. Further, by Lemma 1, and the fact that, in the range of integration,

$$1 \le \left| \frac{\sin\frac{1}{2}\mu^{-1}(\sigma+\tau)}{\sin\frac{1}{2}\mu^{-1}(\sigma-\tau)} \right| \le \left| \frac{\sigma+\tau}{\sigma-\tau} \right|$$

(which is itself a consequence of

$$\frac{\sin\frac{1}{2}\mu^{-1}(\sigma+\tau)}{\sigma+\tau} \le \frac{\sin\frac{1}{2}\mu^{-1}(\sigma-\tau)}{\sigma-\tau}$$

i.e. of the monotonicity of $\sin u/u$), we can bound the integrand in (7.1) by

$$\frac{K}{\tau} \log \left| \frac{\sigma+\tau}{\sigma-\tau} \right| ,$$

and so let $\mu \to \infty$ and apply the dominated convergence theorem to see that, for $\sigma > 0$, ϕ^* satisfies

$$\phi^*(\sigma) = \frac{1}{3\pi} \int_0^\infty \frac{\sin\phi^*(\tau)}{1+\int_0^\tau \sin\phi^*(v)\,dv} \log \left| \frac{\sigma+\tau}{\sigma-\tau} \right| d\tau. \tag{7.2}$$

In fact, since (7.1) implies that $\phi^*_\mu(0) = 0$ for all μ, (7.2) holds also for $\sigma = 0$, and so for $\sigma \ge 0$.

In order to prove Theorem 3, it is only necessary to show that any solution of (7.2), or at least any solution satisfying whatever conditions can be deduced from the limit process $\phi^* = \lim \phi^*_\mu$, cannot satisfy the inequality $0 \le \phi^* \le \frac{1}{6}\pi$. For if the inequality is broken by ϕ^*, then it must be broken by ϕ^*_μ for μ sufficiently large, and the theorem is complete.

To show that $0 \le \phi^* \le \frac{1}{6}\pi$ is impossible, we remark first that we can assert that

$$\left| \int_1^\eta \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma} d\sigma \right| \le K ,$$

where K is independent of η as $\eta \to \infty$. This result is comparable to Lemma 4, and is proved by manipulations on (7.2) which are sufficiently similar to those used in the proof of Lemma 4 as to require no further mention. Now suppose for contradiction that

$$0 \le \phi^* \le \frac{1}{6}\pi . \tag{7.3}$$

Then clearly

$$\int_1^\infty \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma} d\sigma \tag{7.4}$$

exists, and this certainly implies that $\frac{1}{6}\pi$ is one limiting value of $\phi^*(\sigma)$ as $\sigma \to \infty$.

In fact,

$$\phi^*(\sigma) \to \frac{1}{6}\pi \quad \text{as} \quad \sigma \to \infty . \tag{7.5}$$

For if not, suppose that

$$\liminf_{\sigma \to \infty} \phi^*(\sigma) = \frac{1}{6}\pi - 3\delta \qquad (\delta > 0).$$

Then Lemma 3 assures us that, for $\sigma_2 > \sigma_1$,

$$\left| [\phi]_{\sigma_1}^{\sigma_2} \right| \leq K \log \frac{\sigma_2}{\sigma_1} ,$$

and if we choose σ_1 so that $\phi^*(\sigma_1) = \frac{1}{6}\pi - 2\delta$ and σ_2 so that

$$K \log \frac{\sigma_2}{\sigma_1} = \delta,$$

then we see that

$$\phi^*(\sigma) \leq \frac{1}{6}\pi - \delta \qquad \text{for } \sigma_1 \leq \sigma \leq \sigma_2 ,$$

and

$$\left| \int_{\sigma_1}^{\sigma_2} \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma} d\sigma \right| \geq \delta \log \frac{\sigma_2}{\sigma_1} = \frac{\delta^2}{K} .$$

Since this is true for arbitrarily large values of σ_1, σ_2, it contradicts the convergence of (7.4) and so establishes (7.5) (always under the assumption that (7.3) is true).

We can now use the fact (cf. §2) that

$$\frac{1}{6}\pi = \frac{1}{3\pi} \int_0^\infty \frac{1}{\tau} \log \left| \frac{\sigma+\tau}{\sigma-\tau} \right| d\tau$$

to see that

$$\begin{aligned} \phi^*(\sigma) - \frac{1}{6}\pi &= \frac{1}{3\pi} \int_0^\infty \left(\frac{\sin\phi^*(\tau)}{1+\int_0^\tau \sin\phi^*(v)\,dv} - \frac{1}{\tau} \right) \log \left| \frac{\sigma+\tau}{\sigma-\tau} \right| d\tau \\ &= \frac{\sigma}{3\pi} \int_0^\infty \left(\frac{\sin\phi^*(\sigma u)}{1+\int_0^{\sigma u} \sin\phi^*(v)\,dv} - \frac{1}{\sigma u} \right) \log \left| \frac{1+u}{1-u} \right| du , \end{aligned} \tag{7.6}$$

and so

$$\begin{aligned} \int_\rho^\infty \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma} d\sigma &= \frac{1}{3\pi} \int_0^\infty \frac{1}{u} \log \left| \frac{1+u}{1-u} \right| \left\{ \int_{\rho u}^\infty \left(\frac{\sin\phi^*(\sigma u)}{1+\int_0^{\sigma u} \sin\phi^*(v)\,dv} - \frac{1}{\sigma u} \right) d(\sigma u) \right\} du \\ &= -\frac{1}{3\pi} \int_0^\infty \frac{1}{u} \log \left| \frac{1+u}{1-u} \right| \log \left(\frac{1 + \int_0^{\rho u} \sin\phi^*(v)\,dv}{\frac{1}{2}\rho u} \right) du , \end{aligned}$$

where, to obtain the last line, we have used the fact that $\sin\phi^*(\sigma) \to \frac{1}{2}$ as $\sigma \to \infty$.

Finally, therefore, we have

$$(7.7) \qquad \int_\rho^\infty \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma}\, d\sigma$$

$$= -\frac{1}{3\pi}\int_0^\infty \frac{1}{u} \log\left|\frac{1+u}{1-u}\right| \log\left[1 + \frac{1 + \int_0^{\rho u}(\sin\phi^*(v) - \frac{1}{2})dv}{\frac{1}{2}\rho u}\right] du$$

$$= -\frac{1}{3\pi}\int_0^\infty \frac{1}{t} \log\left|\frac{\rho+t}{\rho-t}\right| \log\left[1 + \frac{1 + \int_0^{t}(\sin\phi^*(v) - \frac{1}{2})dv}{\frac{1}{2}t}\right] dt.$$

Since $0 \leq \phi^* \leq \frac{1}{6}\pi$, we know that

$$(7.8) \qquad \int_0^t (\sin\phi^*(v) - \tfrac{1}{2})dv$$

either converges as $t \to \infty$ or else diverges to $-\infty$. In the latter case we can split the last integral as

$$(7.9) \qquad -\frac{1}{3\pi}\left(\int_0^A + \int_A^\infty\right) \frac{1}{t} \log\left|\frac{\rho+t}{\rho-t}\right| \log\left[1 + \frac{1 + \int_0^{t}(\sin\phi^*(v) - \frac{1}{2})dv}{\frac{1}{2}t}\right] = I_1 + I_2,$$

say, where A is a number chosen so that the logarithm is negative for $t \geq A$. Now, for large ρ, in I_1,

$$\log\left|\frac{\rho+t}{\rho-t}\right| \leq K\frac{t}{\rho},$$

and so

$$I_1 = O(\rho^{-1}).$$

Also, (7.5) implies that

$$\frac{1}{t}\int_0^t (\sin\phi^*(v) - \tfrac{1}{2})dv$$

is small for large t, and so

$$I_2 \geq K \int_A^\infty \frac{1}{t^2} \left| \int_0^t (\sin\phi^*(v) - \tfrac{1}{2})dv \right| \log\left|\frac{\rho+t}{\rho-t}\right| dt$$

$$\geq K \int_A^\infty \frac{1}{t^2} \log\left|\frac{\rho+t}{\rho-t}\right| dt$$

$$= \frac{K}{\rho} \int_{A/\rho}^\infty \frac{1}{u^2} \log\left|\frac{1+u}{1-u}\right| du$$

$$\geq K \rho^{-1} \log \rho ,$$

since the integrand behaves like u^{-1} for small u. This implies that (7.9) is positive for large ρ, which contradicts the fact that it is equal to (7.7), and this contradiction establishes that (7.8) converges as $t \to \infty$. Indeed, exactly the same argument shows that (7.8) cannot converge to a limit less than -1, and so, for all $t \geq 0$,

$$(7.10) \qquad 1 + \int_0^t (\sin\phi^*(v) - \tfrac{1}{2})dv \geq 0 .$$

The convergence of (7.8) implies the convergence of

$$\int_0^\infty \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma^{\alpha+1}} d\sigma$$

for any α with $-1 < \alpha < 0$, and in fact

$$(7.11) \qquad \int_0^\infty \frac{\phi^*(\sigma) - \frac{1}{6}\pi}{\sigma^{\alpha+1}} d\sigma$$

$$(7.12) \qquad = \frac{1}{3\pi} \int_0^\infty \left(\frac{\sin\phi^*(\tau)}{1 + \int_0^\tau \sin\phi^*(v)dv} - \frac{1}{\tau} \right) \left\{ \int_0^\infty \frac{1}{\sigma^{\alpha+1}} \log\left|\frac{\sigma+\tau}{\sigma-\tau}\right| d\sigma \right\} d\tau$$

$$(7.13) \qquad = \frac{1}{3} \tan(\tfrac{1}{2}\pi\alpha) \int_0^\infty \frac{1}{\tau^{\alpha+1}} \log \left(1 + \frac{1 + \int_0^\tau (\sin\phi^*(v) - \frac{1}{2})dv}{\frac{1}{2}\tau} \right) d\tau ,$$

by setting $\sigma = \tau u$ in the inner integral in (7.12), using

$$\int_0^\infty \frac{1}{u^{\alpha+1}} \log\left|\frac{1+u}{1-u}\right| du = \frac{\pi}{\alpha} \tan(\tfrac{1}{2}\pi\alpha),$$

and then integrating the outer integral by parts. If we now let $\alpha \to -1$, the integral (7.11) remains bounded because of the convergence of (7.8), and so also therefore does (7.13). Since $\tan(\frac{1}{2}\pi\alpha) \to -\infty$, the integral in (7.13) must tend to 0, whereas in fact the integral has a strictly positive limit because of (7.10). This final contradiction

shows that (7.3) is false and proves the theorem.

This shows that $\phi^* = \frac{1}{6}\pi$ has at least one root, and so also therefore has $\phi^*_\mu = \frac{1}{6}\pi$ for μ sufficiently large. To show that the number of roots of $\phi^*_\mu = \frac{1}{6}\pi$ is unbounded as $\mu \to \infty$, we should need to show that ϕ^* oscillates about $\frac{1}{6}\pi$ infinitely often, and this would be proved by obtaining a contradiction to the fact that $\phi^* - \frac{1}{6}\pi$ is ultimately of one sign. The assumption that $\phi^* - \frac{1}{6}\pi$ is ultimately of one sign allows us to follow through much of the above analysis. We can conclude (7.4) and (7.5), and the convergence of (7.8). One can even show (what we could have shown above but did not need) that

$$\int_0^\infty (\sin\phi^*(v) - \tfrac{1}{2})\,dv = -1 \;. \tag{7.14}$$

But the argument above did require in the last step the inequality (7.10), and this we no longer have if we are merely assuming that $\phi^* - \frac{1}{6}\pi$ is ultimately of one sign; nor does it seem easy to modify the argument.

An alternative approach (under the assumption that $\phi^* - \frac{1}{6}\pi$ is ultimately of one sign) is to use the consequent fact that $\phi^* \to \frac{1}{6}\pi$ to generate an asymptotic expansion and deduce from this the contradiction that ϕ^* must oscillate about $\frac{1}{6}\pi$. Formally, this is easy. For suppose that

$$\phi^*(\sigma) - \frac{1}{6}\pi \sim \sigma^{-\alpha} \quad \text{as} \quad \sigma \to \infty \,,$$

where $\operatorname{re} \alpha > 0$. We have from (7.6) and (7.14) that

$$\sigma^{-\alpha} \sim \frac{1}{3\pi}\int_0^\infty \frac{\tau(\sin\phi^*(\tau)-\frac{1}{2}) + \int_\tau^\infty(\sin\phi^*(v)-\frac{1}{2})\,dv}{\tau(\frac{1}{2}\tau - \int_\tau^\infty(\sin\phi^*(v) - \frac{1}{2})\,dv)} \log\left|\frac{\sigma+\tau}{\sigma-\tau}\right| d\tau$$

$$\sim \frac{1}{3\pi}\int_0^\infty \frac{\frac{1}{2}\sqrt{3}(\tau^{1-\alpha} - \frac{1}{1-\alpha}\tau^{1-\alpha})}{\frac{1}{2}\tau^2} \log\left|\frac{\sigma+\tau}{\sigma-\tau}\right| d\tau$$

$$= -\frac{1}{\pi\sqrt{3}}\,\frac{\alpha}{1-\alpha}\int_0^\infty \frac{1}{\tau^{\alpha+1}} \log\left|\frac{\sigma+\tau}{\sigma-\tau}\right| d\tau$$

$$= -\frac{1}{\sqrt{3}}\,\frac{\sigma^{-\alpha}}{1-\alpha}\tan(\tfrac{1}{2}\pi\alpha),$$

so that α satisfies

$$\tan(\tfrac{1}{2}\pi\alpha) = (\alpha-1)\sqrt{3}, \tag{7.15}$$

and the question is to determine the roots of this equation with smallest positive real part. It is straightworward to verify that (7.15) has no real root satisfying $0 < \alpha \leq 2$ but it does have roots in the strip $0 < \operatorname{re} \alpha < 2$. To see this, apply Rouché's theorem to

$$(\alpha-1)^2\sqrt{3} - (\alpha-1)\tan(\tfrac{1}{2}\pi\alpha).$$

On the line $\operatorname{re}\alpha = 0$,

$$|\tan(\tfrac{1}{2}\pi\alpha)| = |\tanh(\tfrac{1}{2}\pi i\alpha)| \leq 1,$$

so that

$$|(\alpha-1)^2\sqrt{3}| > |(\alpha-1)\tan(\tfrac{1}{2}\pi\alpha)|,$$

and there is a similar argument on $\operatorname{re}\alpha = 2$. Hence (7.15) has precisely two zeros in the strip, which are of course complex conjugates, and it is easy to check that they in fact lie on the line $\operatorname{re}\alpha = 1$. Since the roots of (7.15) with smallest positive real part have non-zero imaginary parts, the function ϕ^* is oscillatory, as required.

The above argument is, however, only formal, and we will not attempt here to make it rigorous.

8. The proof of Lemma 3

It should be remarked that Lemma 3 states much less than is true. The fact that ϕ_μ is continuously differentiable, indeed analytic, on $[0,\pi]$ can be regarded as a particular case of the theorem by Lewy [9] on the analyticity of free boundaries away from stagnation points, and we could also get that information, and estimates on higher derivatives, by a more detailed examination of the proof below. The same applies to ϕ on $(0,\pi]$, but we will in fact restrict ourselves to proving the lemma as stated, since that is all that we require subsequently.

We will prove the result for ϕ. The reader will find that it is even easier for ϕ_μ, and that in that case the relevant estimates are independent of μ, as required. We write

$$\phi(s) = \frac{1}{3\pi}\int_0^\pi\left\{\frac{t\sin\phi(t)}{\int_0^t\sin\phi(u)\,du} - \frac{s\sin\phi(s)}{\int_0^s\sin\phi(u)\,du}\right\}\frac{1}{t}\log\left|\frac{s+t}{s-t}\right|dt$$

$$+\frac{1}{3\pi}\int_0^\pi\frac{t\sin\phi(t)}{\int_0^t\sin\phi(u)\,du}\,\frac{1}{t}\left\{\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| - \log\left|\frac{s+t}{s-t}\right|\right\}dt$$

$$+\frac{1}{3\pi}\,\frac{s\sin\phi(s)}{\int_0^s\sin\phi(u)\,du}\int_0^\pi\frac{1}{t}\log\left|\frac{s+t}{s-t}\right|dt\,,$$

and so

$$\phi'(s) = -\frac{2}{3\pi}\int_0^\pi\left\{\frac{t\sin\phi(t)}{\int_0^t\sin\phi(u)\,du} - \frac{s\sin\phi(s)}{\int_0^s\sin\phi(u)\,du}\right\}\frac{dt}{s^2-t^2}$$

$$(8.1)\qquad +\frac{1}{3\pi}\int_0^\pi\frac{t\sin\phi(t)}{\int_0^t\sin\phi(u)\,du}\,\frac{1}{t}\,\frac{d}{ds}\left\{\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| - \log\left|\frac{s+t}{s-t}\right|\right\}dt$$

$$+\frac{1}{3\pi}\,\frac{s\sin\phi(s)}{\int_0^s\sin\phi(u)\,du}\,\frac{d}{ds}\left\{\int_0^{\pi/s}\frac{1}{v}\log\left|\frac{1+v}{1-v}\right|dv\right\}.$$

There is no difficulty in justifying the above differentiation (for $s > 0$) except for the first integral, since the expression $\{\cdots\}$ in the second term has no singularity at $s = t$; the differentiation of the first integral is also justifiable since ϕ is

Hölder-continuous of order λ , say, this following from the remarks at the close of Lemma 2.

To obtain the estimate

$$s|\phi'(s)| \leq K ,$$

we observe that the third term in (8.1) is $O(1)$ as $s \downarrow 0$, and so causes no trouble. The regular behaviour of the expression $\{\cdots\}$ in the second term of (8.1) assures us that the integrand is bounded for small s and t (which is all that we are really concerned with), and thus again leads to a term $O(1)$ in ϕ' . For small s , therefore, we can write

$$s\phi'(s) = -\frac{2}{3\pi}\int_0^{\pi}\left\{\frac{t\sin\phi(t)}{\int_0^t \sin\phi(u)du} - \frac{s\sin\phi(s)}{\int_0^s \sin\phi(u)du}\right\}\frac{s}{s^2-t^2}\,dt + O(s) \tag{8.2}$$

$$= I_1 + I_2 + I_3 + O(s) ,$$

say, where I_1 is the integral over $[0,k_1s]$, I_2 over $[k_1s,k_2s]$, and I_3 over $[k_2s,\pi]$, $k_1(<1)$ and $k_2(>1)$ being fixed positive numbers which we shall choose more precisely later. Then using

$$\int_0^t \sin\phi(u)du \geq Kt ,$$

obtainable by letting $\mu \to \infty$ in Lemma 1, we obtain

$$|I_1| \leq K\int_0^{k_1s}\frac{dt}{s} \leq K ,$$

and similarly $|I_3| \leq K$.

Also, we can write, for some ξ between s and t ,

$$\frac{t\sin\phi(t)}{\int_0^t \sin\phi(u)du} - \frac{s\sin\phi(s)}{\int_0^s \sin\phi(u)du}$$

$$= (t-s)\left\{\frac{\sin\phi(\xi)}{\int_0^\xi \sin\phi(u)du} + \frac{\xi\cos\phi(\xi)\phi'(\xi)}{\int_0^\xi \sin\phi(u)du} - \frac{\xi\sin^2\phi(\xi)}{(\int_0^\xi \sin\phi(u)du)^2}\right\}$$

and so

$$I_2 \le K \int_{k_1 s}^{k_2 s} \left(\frac{1}{[illegible]} + \frac{[illegible]\,:'([illegible])}{[illegible]} \right) \frac{s}{s+t}\, dt$$

$$\le K + k \sup_{[illegible] \in (0,\infty]} |[illegible]:'([illegible])| ,$$

where k (which depends on k_1 and k_2) can be taken less than unity if k_1 and k_2 are fixed sufficiently close to unity. Thus we have from (8.2) that

$$\sup_{s \in (0,\infty]} |s:'(s)| \le k \sup_{[illegible] \in (0,\infty]} |[illegible]:'([illegible])| + O(1) ,$$

from which the final result of the lemma follows immediately.

9. <u>The proof of Lemma 4</u>

We first make the observation that

$$\left|\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| - \log\left|\frac{s+t}{s-t}\right|\right|$$

$$= \left|\log\left|\frac{\sin\frac{1}{2}(s+t)}{\frac{1}{2}(s+t)}\right| - \log\left|\frac{\sin\frac{1}{2}(s-t)}{\frac{1}{2}(s-t)}\right|\right| \le Kst$$

by an application of the mean value theorem, provided that $s \ge 0$, $t \ge 0$, $s + t \le$ [illegible],

say. Also,

$$\frac{1}{6}\pi = \frac{1}{3\pi}\int_0^\infty \frac{1}{t}\log\left|\frac{s+t}{s-t}\right|dt = \int_0^\pi \frac{1}{t}\log\left|\frac{s+t}{s-t}\right|dt + O(s) ,$$

as $s \downarrow 0$, and so

$$\phi(s) - \frac{1}{6}\pi = \frac{1}{3\pi}\int_0^\pi \left\{\frac{\sin\phi(t)}{\int_0^t \sin\phi(u)\,du} - \frac{1}{t}\right\}\log\left|\frac{s+t}{s-t}\right|dt + O(s), \quad \text{as } s \downarrow 0 ,$$

$$= \frac{s}{3\pi}\int_0^{\pi/s}\left\{\frac{\sin\phi(sv)}{\int_0^{sv}\sin\phi(u)\,du} - \frac{1}{sv}\right\}\log\left|\frac{1+v}{1-v}\right|dv + O(s) .$$

Thus

$$\int_\eta^{\frac{1}{2}\pi}\frac{\phi(s)-\frac{1}{6}\pi}{s}\,ds$$

$$= \frac{1}{3\pi}\int_\eta^{\frac{1}{2}\pi}\left\{\int_0^{\pi/s}\left\{\frac{\sin\phi(sv)}{\int_0^{sv}\sin\phi(u)\,du} - \frac{1}{sv}\right\}\log\left|\frac{1+v}{1-v}\right|dv\right\}ds$$

$$= \frac{1}{3\pi}\int_0^{\pi/\eta}\log\left|\frac{1+v}{1-v}\right|\left\{\int_\eta^{\min(\pi/v,\frac{1}{2}\pi)}\left(\frac{\sin\phi(sv)}{\int_0^{sv}\sin\phi(u)\,du} - \frac{1}{sv}\right)ds\right\}dv$$

$$= \frac{1}{3\pi}\int_0^{\pi/\eta}\frac{1}{v}\log\left|\frac{1+v}{1-v}\right|\left[\log\left(\frac{\int_0^{sv}\sin\phi(u)\,du}{sv}\right)\right]_{s=\eta}^{s=\min(\pi/v,\frac{1}{2}\pi)} dv .$$

But Lemma 1 (in the limit as $\mu \to \infty$) tells us that the logarithm in the last formula is uniformly bounded for all relevant values of sv, and so the integral is bounded by

$$K \int_0^{\pi/\eta} \frac{1}{v} \log\left|\frac{1+v}{1-v}\right| dv ,$$

which gives the required result.

References

1. A. I. Nekrasov, The exact theory of steady state waves on the surface of a heavy liquid, Technical Summary Report No. 813 (1967), Mathematics Research Center, University of Wisconsin (transl. D. V. Thampuran, ed. C.W. Cryer).
2. L. M. Milne-Thomson, Theoretical Hydrodynamics (5th ed., Macmillan, 1968).
3. G. Keady and J. Norbury, On the existence theory for irroattional water waves, Proc. Camb. Phil. Soc. 83 (1978), 137-157.
4. Yu. P. Krasovskii, On the theory of steady-state waves of finite amplitude, U.S.S.R. Comp. Math. and Math. Phys. 1 (1962), 996-1018.
5. J. F. Toland, On the existence of a wave of greatest height and Stokes' conjecture, Proc. Royal Soc. A363 (1978), 469-485.
6. T. Levi-Civita, Détermination rigoureuse des ondes permanentes d'ampleur finie, Math. Ann. 93 (1925), 264-314.
7. G. G. Stokes, Mathematical and Physical Papers (Vol. I, C.U.P., 1880).
8. M. S. Longuet-Higgins and M. J. H. Fox, Theory of the almost-highest wave: the inner solution, J. Fluid Mech. 80 (1977), 721-741.
9. H. Lewy, A note on harmonic functions and a hydrodynamical application, Proc. Amer. Math. Soc. 3 (1952) 111-113.

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The integral equation

$$\phi_\mu(s) = \frac{1}{3\pi}\int_0^\pi \frac{\sin\phi_\mu(t)}{\mu^{-1}+\int_0^t \sin\phi_\mu(u)du} \log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| dt$$

was derived by Nekrasov to describe waves of permanent form on the surface of a non-viscous, irrotational, infinitely deep flow, the function ϕ_μ giving the angle which the wave surface makes with the horizontal. The wave of greatest height is the singular case $\mu \to \infty$, and it is shown that there exists a solution

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20. Abstract (continued)

ϕ_∞ to the equation in this case and that it can be obtained as the limit (in a specified sense) as $\mu \to \infty$ of solutions for finite μ.

Stokes conjectured that $\phi_\infty(s) \to \frac{1}{6}\pi$ as $s \downarrow 0$, so that the wave is sharply crested in the limit case; and Krasovskii conjectured that $\sup_{s\in[0,\pi]} \phi_\mu(s) \leq \frac{1}{6}\pi$ for all finite μ. While the present paper makes only limited progress towards deciding Stokes' conjecture, Krasovskii's conjecture is shown to be false for sufficiently large μ, the angle exceeding $\frac{1}{6}\pi$ in what is a boundary layer.